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AN OPERATOR ON L^p WITHOUT BEST COMPACT APPROXIMATION†

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ABSTRACT

We construct, for $1 < p < \infty$, $p \neq 2$, an operator on L^p whose distance to the space of compact operators on L^p is not attained. We also show that the identity operator on L^p, $p \neq 1, 2, \infty$ has a unique best compact approximation.

Introduction

Let X be a Banach space, and denote by $B(X)$ and $K(X)$ the spaces of all bounded, respectively compact, linear operators on X with the operator norm. Several authors studied the problem of identifying spaces X so that for each $T \in B(X)$, its distance to $K(X)$ is attained. Of particular interest is the problem of identifying which of the standard Banach spaces have this property. Several authors proved that $X = l_p$ $(1 \leqq p < \infty)$ has this property (see [1] and its references). It was shown by Feder [4] that when X is L^1, L^∞, l^∞ or $C[0,1]$, there are operators on X without best compact approximation (see also [2]). For L^p, $p \neq 1, 2, \infty$, only partial results were known; Weis [6] showed that certain integral operators on L^p do have a best compact approximation. In this article we show that this is not true for all operators on L^p:

THEOREM 1. *For every $1 \leqq p \leqq \infty$, $p \neq 2$, there is an operator on L^p whose distance to the compact operators on L^p is not attained.*

As mentioned above, the theorem is known for $p = 1, \infty$, and fails for $p = 2$. We shall thus assume throughout that $1 < p < \infty$ and $p \neq 2$.

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In §1 we describe a simple general successive approximation scheme, which was used in all the positive results described above. The key observation is that for the particular case at hand, the failure of this scheme is *equivalent* to the failure of best compact approximation for some operator on L^p. In §2 we prove Theorem 1 by showing that the successive approximation scheme indeed fails for operators on L^p, $p \neq 1,2,\infty$.

In §3 we show that L^p behaves differently from l_p also with respect to the uniqueness of best compact approximation. If $T \in B(l_p)$ is not compact, its distance to $K(l_p)$ is attained at many compact operators. This is a general phenomenon in M-ideals [5]; see also [1]. This is no longer true for operators on L^p. As the next theorem shows, the zero operator on L^p is the unique best compact approximation of the identity operator, I, in a very strong sense.

THEOREM 2. *Fix $p \neq 1,2,\infty$. For each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ so that if $K \in K(L^p)$ and $\|K\| > \varepsilon$, then $\|I - K\| \geqq 1 + \delta(\varepsilon)$.*

We shall use standard Banach space notation and terminology; see e.g. [3].

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§1. Let E be a Banach space and $F \subset E$ a closed subspace. We shall say that E and F satisfy the successive approximation scheme if there is a function $\varphi(\varepsilon) > 0$, defined for $\varepsilon > 0$ and satisfying $\lim_{\varepsilon \to 0+} \varphi(\varepsilon) = 0$, with the following property:

Given any $x \in E$ with $1 = d(x, F) \leqq \|x\| < 1 + \varepsilon$, and any $\delta > 0$, there is a $y \in F$ with $\|x - y\| < 1 + \delta$ and so that $\|y\| < \varphi(\varepsilon)$.

LEMMA 1. *If E and F satisfy the successive approximation scheme, then for every $x \in E$ there is a $y \in F$ with $d(x, F) = \|x - y\|$.*

PROOF. Assume $d(x, F) = 1$, and for every j find $\delta_j > 0$ so that $\varphi(\delta_j) < 2^{-j}$. Let $y_1 \in F$ be such that $\|x - y_1\| < 1 + \delta_1$. Now find $y_2 \in F$ so that $\|y_2\| < \varphi(\delta_1) < 2^{-1}$ ad so that $\|x - y_1 - y_2\| < 1 + \delta_2$. Continue inductively: If $y_1, \ldots, y_j$ are already chosen with $\|x - \Sigma_1^j y_i\| < 1 + \delta_j$, choose $y_{j+1} \in F$, $\|y_{j+1}\| < \varphi(\delta_j) < 2^{-j}$ and so that $\|x - \Sigma_1^{j+1} y_i\| < 1 + \delta_{j+1}$. Put now $y = \Sigma_1^\infty y_j$. The series is absolutely convergent, and obviously $\|x - y\| = 1 = d(x, F)$.

Before we pass to the converse for the space of operators on L^p, we introduce some standard notation. We denote by $(\Sigma \oplus L^p)_p$ the space of all sequences $f = (f_1, f_2, \ldots)$ with the norm $\|f\| = (\Sigma \|f_i\|_p^p)^{1/p}$. The space $(\Sigma \oplus L^p)_p$ is isometric to L^p. If T_n is a uniformly bounded sequence of operators on L^p, we define $T = \oplus T_n$ on $(\Sigma \oplus L^p)_p$ by $T(f_1, f_2, \ldots) = (T_1 f_1, T_2 f_2, \ldots)$. Then T is a bounded linear operator with $\|T\| = \sup \|T_n\|$. Let P_n be the projection on the nth copy of L^p, i.e. $P_n(f_1, f_2, \dots) = f_n$, then $\|P_n\| = 1$. If K is a compact operator on $(\Sigma \oplus L^p)_p$ then P_nKP_n is a compact operator on L^p, and $\|P_nKP_n\| \to 0$. Indeed, otherwise we could find $\alpha > 0$, an increasing sequence n_j, and $f_j \in L^p$ with $\|f_j\| = 1$ and $\|P_{n_j}KP_{n_j}f_j\| \geqq \alpha$. But then putting $g_j = (0, \dots, 0, f_j, 0, \dots)$ (f_j in the n_jth position) we have $\|g_j\| = 1$, $g_j \xrightarrow{\omega} 0$ in $(\Sigma \oplus L^p)_p$, yet $\|Kg_j\| \geqq \|P_{n_j}KP_{n_j}g_j\| \geqq \alpha$, contradicting the compactness of K.

LEMMA 2. *If the spaces $E = B(L^p)$ and $F = K(L^p)$ fail the successive approximation scheme, then there is an operator $T \in B(L^p)$ with no best compact approximation.*

PROOF. Assume the scheme fails, and find $\alpha > 0$, a sequence of operators T_n on L^p and numbers $\delta_n > 0$ so that

(i) $1 = d(T_n, K(L^p)) \leqq \|T_n\| < 1 + 1/n$.

(ii) If $K \in K(L^p)$ and $\|T_n - K\| < 1 + \delta_n$, then $\|K\| \geqq \alpha$.

We show that $T = \bigoplus T_n$ has no best compact approximation. Note first that $d(T, K((\Sigma \oplus L^p)_p)) \leqq 1$. Indeed, given $\varepsilon > 0$, fix $N > 1/\varepsilon$ and find for each $n \leqq N$ a compact operator K_n on L^p with $\|K_n - T_n\| < 1 + \varepsilon$. Put also $K_n = 0$ for $n > N$, then $K = \bigoplus K_n$ is compact and $\|T - K\| = \sup \|T_n - K_n\| < 1 + \varepsilon$.

Assume now that there is a compact operator K on $(\Sigma \oplus L^p)_p$ with $\|T - K\| \leqq 1$. Since K is compact we can find n so large that $\|P_nKP_n\| < \alpha/2$. But then

$$\|T_n - P_nKP_n\| = \|P_n(T - K)P_n\| \leqq \|T - K\| \leqq 1 < 1 + \delta_n$$

yet P_nKP_n is a compact operator on L^p with $\|P_nKP_n\| < \alpha/2$. This contradicts (ii).

§2. From now on fix $2 < p < \infty$. (For $1 < p < 2$ the result follows by duality.) Rather than work with $L^p[0,1]$, it will be more convenient for us to work in this section with the space $L^p(R)$, where R is the rectangle $R = [0,2] \times [0,1]$ with the Lebesgue measure μ (i.e. $\mu(R) = 2$). Of course $L^p(R)$ is isometric to L^p. Let P be the projection in $L^p(R)$ on the space of all functions which depend only on their x-coordinate, i.e.

$$(Pf)(x, y) = \int_0^1 f(x, t)dt.$$

P is a norm one projection on an infinite dimensional subspace, hence

$$d(P, K(L^p(R))) = \|P\| = 1.$$

Fix any $0 < \varepsilon < 1$. We shall construct a rank one operator S on $L^p(R)$ so that

$\|P+S\| \leqq 1+\varepsilon^{1/p}+\varepsilon^{1/q}$ (where $p^{-1}+q^{-1}=1$), and so that whenever K is any compact operator on $L^p(R)$ with $\|K\|<1/4$, then

$$(*)\qquad \|P+S-K\| \geqq 1+C\varepsilon$$

for some positive constant C depending only on p. Since $d(P+S, K(L^p(R))) = d(P, K(L^p(R))) = 1$ and since ε is arbitrary, this shows that the successive approximation scheme fails, and Theorem 1 will follow from Lemma 2.

THE OPERATOR S. Let $A=[0,1]\times[0,\varepsilon]$ and $B=[1,2]\times[0,\varepsilon]$, and let χ_A and χ_B be their indicator functions. Define

$$Sf=\varepsilon^{-1}\left(\int_B f d\mu\right)\chi_A.$$

Then $S\chi_B=\chi_A$, and using Hölder's inequality one obtains that $\|S\|=1$.

To estimate $\|P+S\|$, fix $f\in L^p(R)$, and let $f_1=f\,|_B$ and $f_2=f-f_1$, and we make some preliminary computations.

$Pf_1(x,y)=0$ if $x\leqq 1$, and when $x\geqq 1$ we have

$$|Pf_1(x,y)|=\left|\int_0^\varepsilon f_1(x,t)dt\right| \leqq \varepsilon^{1/q}\left(\int_0^\varepsilon |f_1(x,t)|^p dt\right)^{1/p}.$$

Thus $\|Pf_1\|^p \leqq \varepsilon^{p/q}\int_B |f|^p d\mu$, hence

(i) $\|Pf_1\| \leqq \varepsilon^{1/q}\|f\|$.

Since Pf_2 is constant on vertical segments, we have

(ii) $\|\chi_A Pf_2\| \leqq \varepsilon^{1/p}\|Pf_2\| \leqq \varepsilon^{1/p}\|f_2\| \leqq \varepsilon^{1/p}\|f\|$.

Finally Sf_1 and $(1-\chi_A)Pf_2$ are disjointly supported, hence

(iii) $\|Sf_1+(1-\chi_A)Pf_2\| = (\|Sf_1\|^p+\|(1-\chi_A)Pf_2\|^p)^{1/p} \leqq (\|f_1\|^p+\|f_2\|^p)^{1/p} = \|f\|$.

Noting that $Sf_2=0$, we use (i)–(iii) and obtain

$$\|(P+S)f\| \leqq \|Pf_1\|+\|\chi_A Pf_2\|+\|Sf_1+(1-\chi_A)Pf_2\| \leqq (\varepsilon^{1/q}+\varepsilon^{1/p}+1)\|f\|.$$

PROOF OF THEOREM 1. Let K be any compact operator on $L^p(R)$. The idea of the proof is that the only way for $\|P+S-K\|$ to be small is if K "cancels" what S did, i.e. $K\chi_B$ should be approximately χ_A, but this forces $\|K\|$ to be large. We now formalize this.

Assume K is a compact operator with $\|K\|\leqq 1/4$. Let r_n be the Rademacher functions on $[0,1]$ and define $h_n\in L^p(R)$ by

$$h_n(x,y)=\begin{cases} r_n(x), & 0\leqq x\leqq 1,\\ 0, & 1<x\leqq 2.\end{cases}$$

The functions h_n are disjointly supported from χ_B, hence for each number λ we have

$$\| h_n + \lambda \chi_B \| = (\| h_n \|^p + |\lambda|^p \| \chi_B \|^p)^{1/p} = (1 + \varepsilon |\lambda|^p)^{1/p}.$$

We shall show that

$$(**) \qquad \lim_n \|(P + S - K)(h_n + \lambda \chi_B)\| \geqq (1 + 3\varepsilon |\lambda|^2/16)^{1/p}.$$

Choosing λ so that $\varepsilon |\lambda|^2/16 = \varepsilon |\lambda|^p$, and using the fact that

$$\left(\frac{1+3x}{1+x}\right)^{1/p} \geqq 1 + x/p \qquad \text{for small positive } x,$$

we obtain that

$$\| P + S - K \| \geqq (1 + 3\varepsilon |\lambda|^2/16)^{1/p} (1 + \varepsilon |\lambda|^p)^{-1/p} \geqq 1 + \varepsilon |\lambda|^p / p = 1 + C\varepsilon$$

where $C = |\lambda|^p/p$, proving (*).

Since $\|K\| < 1/4$, $\|K(\chi_B)\| \leqq \|\chi_B\|/4 = \varepsilon^{1/p}/4$. By Chebyshev's inequality we obtain that

$$\mu\{(x, y): |K(\chi_B)| > 1/2\} \leqq 2^p \|K(\chi_B)\|^p \leqq 2^{-p}\varepsilon < \varepsilon/4.$$

Thus, if we put $D = \{(x, y) \in A : |K(\chi_B)(x, y)| \leqq 1/2\}$, then $\mu(D) \geqq \mu(A) - \varepsilon/4 = 3\varepsilon/4$.

Put $I = [0,1] \times [0,1]$, the unit square, and let $f = (S - K)\chi_B = \chi_A - K(\chi_B)$. By the above $f \geqq 1/2$ on D, a subset of I of measure at least $3\varepsilon/4$, and thus

$$\int_I |f|^2 d\mu \geqq \int_D |f|^2 d\mu \geqq 3\varepsilon/16.$$

Now $h_n \xrightarrow{\omega} 0$ and $S - K$ is compact, so $\lim \|(S - K)h_n\| = 0$. Since also $P\chi_B \big|_I \equiv 0$ and $Ph_n = h_n$ we see that

$$\begin{aligned}
\lim \|(P + S - K)(h_n + \lambda \chi_B)\| &= \lim \| h_n + \lambda (f + P\chi_B)\| \\
&\geqq \lim \left(\int_I |h_n + \lambda f|^p d\mu \right)^{1/p} \\
&\geqq \lim \left(\int_I |h_n + \lambda f|^2 d\mu \right)^{1/2} \\
&= \left(1 + |\lambda|^2 \int_I |f|^2 d\mu \right)^{1/2}
\end{aligned}$$

(because $\int_I |h_n|^2 d\mu = 1$ and $\lim \int_I f h_n d\mu = 0$ because $h_n \xrightarrow{\omega} 0$).

But $\int_I |f|^2 d\mu \geqq 3\varepsilon/16$, thus, since $p > 2$ we have

$$\lim \|(P+S-K)(h_n+\lambda\chi_B)\| \geqq (1+3\varepsilon\,|\lambda|^2/16)^{1/2} \geqq (1+3\varepsilon\,|\lambda|^2/16)^{1/p}$$

which proves (**) which, in turn, implied (*), hence the theorem.

§3. In this section we again work with $L^p = L^p[0,1]$. We fix $2 < p < \infty$ (again the case $1 < p < 2$ follows by duality). We shall need two lemmas.

LEMMA 3. *There are positive constants β and α, depending only on p, and a sequence $h_n \in L^p$ so that*
(i) $\|h_n\| \leqq \alpha$ *for all n,*
(ii) $h_n \xrightarrow{\omega} 0$,
(iii) $\|1-h_n\|^p + \beta|1-\lambda|^p \leqq \|\lambda - h_n\|^p$ *for all n and for all scalars λ.*

PROOF. Let X be the one-dimensional subspace of constants in L^p, and let $f = \chi_{[0,2/3]} - 2\chi_{[2/3,1]}$. Let $a \in X$ be the nearest point in X to f. It is easy to check that $a \neq 0$. Let $h = f/a$, then the nearest point to h in X is the constant 1. We put $\alpha = \|h\|$. Let h_n be a sequence of stochastically independent functions on $[0,1]$ with the same distribution as h. Then $\|h_n\| = \alpha$, and $h_n \xrightarrow{\omega} 0$ because $\int h d\mu = 0$. Also for each constant λ, $\|h_n + \lambda\| = \|h + \lambda\|$, thus to prove (iii) we need only show that it holds for h. We shall use Clarkson's inequality, which for $p > 2$ gives

$$\|\tfrac{1}{2}(f+g)\|^p + \|\tfrac{1}{2}(f-g)\|^p \leqq \tfrac{1}{2}(\|f\|^p + \|g\|^p) \qquad \text{for all } f, g \in L^p.$$

Fix λ, and take $f = 1-h$ and $g = \lambda - h$. Since $\|\frac{1}{2}(f+g)\| = \|\frac{1}{2}(1+\lambda) - h\| \geqq \operatorname{dist}(h, X) = \|1-h\|$, and since $\|f-g\|^p = |1-\lambda|^p$ the lemma follows with $\beta = 2^{1-p}$.

LEMMA 4. *Let α and β be as in Lemma 3. If $f, g \in L^p$, with $\|f\| = 1$, then there are functions $h_n \in L^p$ so that*
(a) $\|h_n\| \leqq \alpha$ *for all n,*
(b) $h_n \xrightarrow{\omega} 0$,
(c) $\|f-h_n\|^p + \frac{1}{2}\beta\|g\|^p \leqq \|f-g-h_n\|^p$ *for all n.*

PROOF. We shall prove the lemma under the additional assumption that both f and g are simple functions, measurable with respect to the partition $\bigcup_1^m A_i$ of $[0,1]$, and also that $f(x) \neq 0$ for all x — but with $\frac{1}{2}\beta$ replaced by β in (c). Since f and g can be approximated arbitrarily well by functions of the above form, this will prove the lemma.

By renormalization and change of variable, Lemma 3 gives that for any subset $A \subset [0,1]$ of positive measure, $a \neq 0$ and b there are functions $h_n \in L^p$, supported in A, $h_n \xrightarrow{\omega} 0$, so that $\|h_n\| \leqq |a| \alpha \mu(A)^{1/p}$ and so that

$$\int_A |a - h_n|^p d\mu + \beta |a - b|^p \mu(A) \leqq \int_A |b - h_n|^p d\mu.$$

Put $a_i = f|_{A_i}$, $b_i = (f-g)|_{A_i}$ and use the above to find h_n^i, supported in A_i satisfying $h_n^i \xrightarrow{\omega} 0$, $\|h_n^i\| \leqq |a_i| \alpha \mu(A_i)^{1/p}$ and

$$\int_{A_i} |a_i - h_n^i|^p d\mu + \beta |a_i - b_i|^p \mu(A_i) \leqq \int_{A_i} |b_i - h_n^i|^p d\mu.$$

Taking $h_n = \Sigma_i h_n^i$ and summing the above inequalities over i gives (c). Condition (b) is obvious, and

$$\|h_n\|^p = \Sigma \|h_n^i\|^p \leqq \Sigma |a_i|^p \alpha^p \mu(A_i) = \alpha^p \|f\|^p = \alpha^p$$

proving (a).

Proof of Theorem 2. Let K be a compact operator on L^p, and assume $\|f\| = 1$ and $\|Kf\| \geqq \varepsilon$. Put $g = Kf$ and use Lemma 4 to find h_n satisfying (a)–(c). Since K is compact and $h_n \xrightarrow{\omega} 0$, $\|Kh_n\| \to 0$, thus by (c)

$$\lim_n \|(I-K)(f-h_n)\|^p = \lim \|f - g - h_n\|^p \geqq \lim(\|f-h_n\|^p + \beta \|g\|^p/2).$$

Thus $\|I-K\|^p \geqq \lim(\|f-h_n\|^p + \beta\|g\|^p/2)/\|f-h_n\|^p$. Since $\|g\| \geqq \varepsilon$ and $\|f-h_n\| \leqq 1+\alpha$ we see that $\|I-K\| \geqq (1+\beta\varepsilon^p(1+\alpha)^{-p}/2)^{1/p} = 1+\delta(\varepsilon)$.

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